A SIMPLE WIND-PUMP ANALYSIS

MICHEL LOUGE

This document outlines a simple power budget for the wind pump, and an analysis of its dynamics.

1. Power analysis

Consider a single cylinder - single acting engine consisting a slider-crank mechanism with connecting rod of length L and crankshaft of radius R moving a single piston of cross-section area A along a straight path measured along the coordinate x from top-dead-center (TDC). The geometry yields

$$x = L + R - [\sqrt{L^2 - R^2 \sin^2\theta} + R\cos\theta], \tag{1}$$

where θ is the crank angle with $\theta = 0$ at TDC. In the limit of large L/R, the motion is nearly sinusoidal with $x \simeq R(1 - \cos\theta)$ and velocity $\dot{x} \simeq R\dot{\theta}\sin\theta$. If the cylinder intakes water with $0 < \theta < \pi$ and exhausts it with $\pi < \theta < 2\pi$, and if the angular velocity is steady ($\dot{\theta} =$ constant), then the mean volume flow rate over an entire cycle is

$$\dot{Q} \simeq -AR\dot{\theta}\frac{1}{2\pi}\int_{\theta=\pi}^{2\pi} \sin\theta d\theta = AR\dot{\theta}/\pi. \tag{2}$$

If the cylinder is immersed horizontally below the water line, the hydrostatic pressure due to this immersion is the same on both sides of the piston, and thus results in no net force. Then, if we neglect fluid inertia, the only force on the piston during the exhaust stroke is constant and equal to $\rho_w ghA$, where h is the required height to lift water above the tank's waterline, and ρ_w is water density. Thus the instantaneous power required to lift water $P = \rho_w ghA\dot{x}$; its average is then

$$\bar{P} = \rho_w gh\dot{Q}, \tag{3}$$

where $\dot{Q}$ is calculated in Eq. (2).

If the crankshaft is connected to a large sprocket of radius R_1, itself linked through a frictionless chain to a smaller sprocket of radius R_0 on the windmill shaft, the angular velocity ω of the windmill is related to that of the crankshaft through

$$R_0\omega = R_1\dot{\theta}. \tag{4}$$

Combining Eqs. (2) – (4), we find

$$\bar{P} = \rho_w gh\Big(\frac{AR}{\pi}\Big)\Big(\frac{R_0}{R_1}\Big)\omega, \tag{5}$$

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which defines the operating load of the wind pump. The intersection of this straight line with the windmill characteristic power curve yields the operating point.

This analysis does not address the dynamics of the system. Doing so would require knowledge of the moment of inertia of the sprockets, shaft and wind blades, as well as the mass of the piston and added mass from the fluid. Coupled dynamical equations involving forces on the piston and torques on the two sprockets could be reduced to a single ODE in θ that would involve the torque curve measured in the lab. The principal benefit of such analysis would be to calculate intermittency of the system that is mainly due to single-single-acting nature of the pump.

2. Dynamics

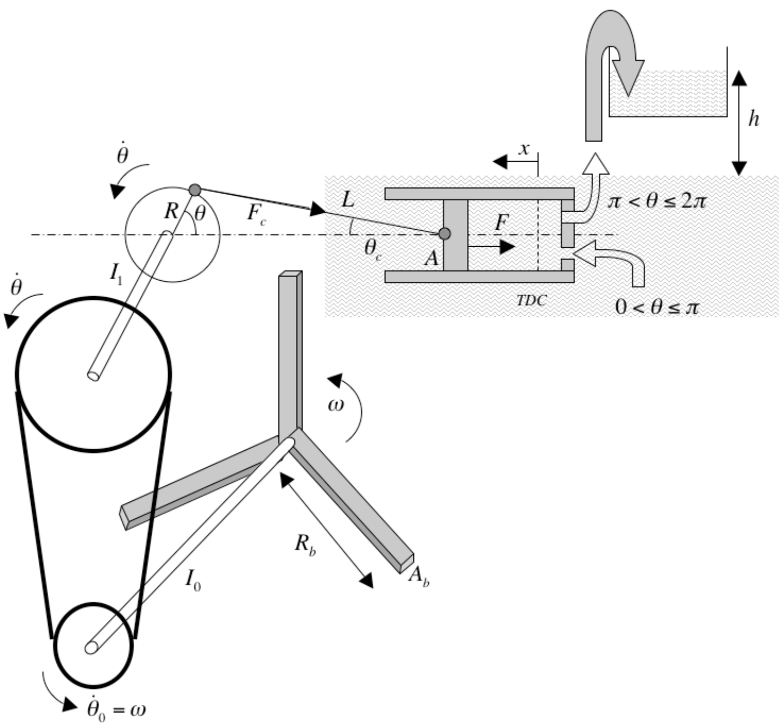

Figure 1. Sketch of the windpump system.

Consider the dynamics of the problem outlined in section 1. Variables and geometries are sketched in Fig. 1. For simplicity, we assume that the chain and piston have negligible inertia and friction. The equation of motion of the large sprocket is

$$I_1\ddot{\theta} = T_1 - F_c R\sin(\theta + \theta_c), \tag{6}$$

where I_1 is the moment of inertia of the large sprocket, T_1 is the torque exerted by the chain on the large sprocket, F_c is the force exerted by the sprocket on the piston along the connecting rod,[1] and θ_c is the angle of the latter with the cylinder axis at the piston. Trigonometry yields $\sin(\theta_c) = \sin(\theta)/(L/R)$. The equation of motion of the piston is

$$m\ddot{x} = F_c \cos(\theta_c) - F, \tag{7}$$

where F is the net force exerted by the piston on water.[2] Assuming that the force is only exerted during exhaust, $F = -\rho_w g h A$ for $\pi < \theta \leqslant 2\pi$ and zero during intake.[3] Neglecting piston mass m, Eq. (7) reduces to the force balance $F = F_c \cos(\theta_c)$. Similarly, the motion of the small sprocket connected by a shaft to the windmill is

$$I_0 \ddot{\theta}_0 = T - T_0, \tag{8}$$

where I_0 is the moment of inertia of the small sprocket, the shaft and the windmill,[4] T is the torque generated by wind on the windmill/shaft/small sprocket assembly, and T_0 is the torque exerted by this assembly on the chain.[5] A simple balance of forces on the chain relates the torques that it transmits

$$\frac{T_0}{R_0} = \frac{T_1}{R_1} \tag{9}$$

Note that $\dot{\theta}_0 \equiv \omega$, where ω is the angular velocity of the windmill. Then, Eq. (4) also relates angular accelerations of the two sprockets,

$$R_1 \ddot{\theta} = R_0 \ddot{\theta}_0. \tag{10}$$

We model windmill torque performance as an inverted parabola (Fig. 2),

$$\frac{T}{T_m} = \left(\frac{\omega}{\omega_m}\right)\left[2 - \left(\frac{\omega}{\omega_m}\right)\right]. \tag{11}$$

Combining the preceding equations to eliminate dependent variables in terms of θ, we find that the dynamics of this system reduce to the single ODE

$$\left[I_1 + I_0\left(\frac{R_1}{R_0}\right)^2\right]\ddot{\theta} = \left(\frac{R_1}{R_0}\right)^2 T_m \frac{\dot{\theta}}{\omega_m}\left[2 - \left(\frac{R_1}{R_0}\right)\frac{\dot{\theta}}{\omega_m}\right] - FR\frac{\sin(\theta + \theta_c)}{\cos(\theta_c)}. \tag{12}$$

We measure time relative to ω_m, define $t^\dagger \equiv t\omega_m$, and write the dimensionless version of Eq. (12)

$$\ddot{\theta}^\dagger = T_m^\dagger \dot{\theta}^\dagger \left[2 - R^\dagger \dot{\theta}^\dagger\right] - \mathcal{L}^\dagger F^\dagger \frac{\sin(\theta + \theta_c)}{\cos(\theta_c)}, \tag{13}$$

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[1]... so the force exerted by the piston on the sprocket is $-F_c$, hence the minus sign in the sprocket dynamic Eq. (6).

[2]... so $-F$ is the force exerted by water on the piston, hence the sign in Eq. (7).

[3]During exhaust, the force on water is directed along the negative x-axis and thus $F < 0$ for $\pi < \theta \leqslant 2\pi$. If there was more than one piston, F would the sum of the contributions of each piston with appropriate phase lags.

[4]If the moment of inertia of the windmill blades is not available, you might estimate them by considering that each blade is a simple parallelepiped of constant cross section A_b, material density ρ_b and length R_b. In this case, the moment of inertia of each blade is $(1/3)\rho_b A_b R_b^3$ Watch for the cubic dependence on R_b!

[5]... so the torque on the small sprocket assembly if $-T_0$, hence the sign in Eq. (8).

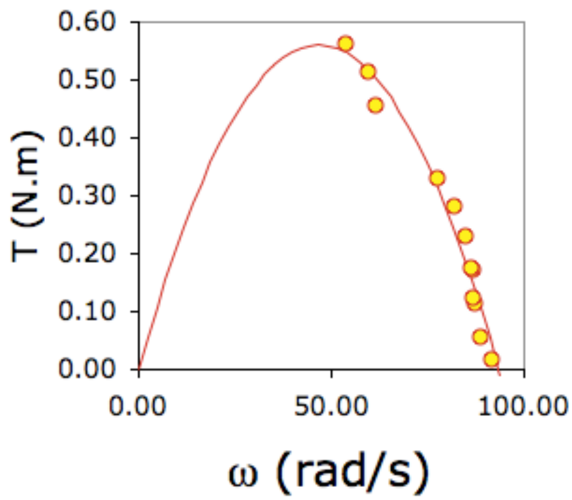

FIGURE 2. Torque characteristics of a windmill. Symbols are data and the line is a least-squares fit to Eq. (11).

where $R^\dagger \equiv R_1/R_0$ and

$$T_m^\dagger \equiv \left(\frac{R_1}{R_0}\right)^2 \frac{T_m}{\omega_m^2[I_1 + I_0(R_1/R_0)^2]} \tag{14}$$

is a characteristic torque of the windmill and

$$\mathcal{L}^\dagger \equiv \frac{\rho_w g h A R}{\omega_m^2[I_1 + I_0(R_1/R_0)^2]} \tag{15}$$

is a characteristic load on the pump. The dimensionless force on the piston is $F^\dagger = -1$ for $\pi < \theta \leqslant 2\pi$ and zero otherwise.[6]

It is evident that, with a vanishing torque at $\omega = 0$, the windpump cannot start unassisted.[7] In this model, we merely assume that it starts with a small angular velocity and at a finite value of θ. The MATLAB program `windpump.m` integrates the equation of motion and finds the dimensionless volume flow rate $\dot{V}^\dagger$ by integrating it, along with θ and $\dot{\theta}^\dagger$, in MATLAB's `ODE45` solver using

$$\frac{dV^\dagger}{dt^\dagger} = \dot{\theta}^\dagger f_2(\theta), \tag{16}$$

where $f_2(\theta)$ is a kinematic function calculated in MATLAB as `pistonkin.m` such that $\dot{x} \equiv 2R\dot{\theta} f_2(\theta; L/R)$. At the program's end, the actual volume flowrate is found using $\dot{V} = 2RA\dot{V}^\dagger\dot{\theta}$.

[6]Note that the term $F^\dagger \sin(\theta + \theta_c)/\cos(\theta_c) > 0$ for $\pi < \theta \leqslant 2\pi$; it is indeed a load on the system, contributing to the deceleration of $\dot{\theta}$ in Eq. (12).

[7]In practice, the torque is not zero at rest.

ODE45 is a versatile program that carries out 4-th order Runge-Kutta integration of the column vector $(\theta; \dot{\theta}^\dagger; V^\dagger)$ simultaneously, whereby

$$\frac{d}{dt^\dagger}\begin{pmatrix}\theta \\ \dot{\theta}^\dagger \\ V^\dagger\end{pmatrix} = \begin{pmatrix}\dot{\theta}^\dagger \\ T_m^\dagger \dot{\theta}^\dagger[2 - R^\dagger \dot{\theta}^\dagger] - \mathcal{L}^\dagger F^\dagger \sin(\theta+\theta_c)/\cos(\theta_c) \\ \dot{\theta}^\dagger f_2(\theta)\end{pmatrix} \tag{17}$$

In `ODE45` integration, we must make sure that Matlab does not automatically choose a time step that exceeds the period of the crankshaft.[8] To that end, we set the maximum time step Δt to a fourth of the piston cycle period that the windmill can muster at its maximum rotation rate $2\omega_m$. Because the windmill period is $2\pi/\omega$, and the (longer) piston cycle period is $(2\pi/\omega)(R_1/R_0)$, we impose the maximum dimensionless tim step is $\Delta t^\dagger < \pi R^\dagger/4$.

As example, the commands:

```
inpI = [.03;.12;.04;.15;pi/4*.04^2;8000*pi/2*.04^4*.01+...
2000*.1*.02*.75^3;8000*pi/2*.15^4*.01;50;.6;1000*9.81] ;
[tiarr,tharr,vearr,vfr,omef,freq,volp] = windpump(inpI,pi/2,3,600);
```

yield the result in Fig. 3. A copy of the Matlab program is available on Blackboard. If your design calls for more than one single-acting, single-cylinder, it is relatively straightforward to amend the program with a torque on the crankshaft that incorporates several pistons with appropriate phase lags.

Finally, Eq. (13) lends itself to an approximate analytical solution. Because oscillations of the slider-crank are typically fast compared to the rise time of the crankshaft angular velocity, we can reasonably assume that $\dot{\theta}^\dagger$ is constant during a cycle of the piston ($0 < \theta \leqslant 2\pi$). We call this new variable $\dot{\bar{\theta}}^\dagger$.

In this case, the second term on the right-hand-side of Eq. (13), which represents the load on the piston that is constant over the interval $\pi < \theta \leqslant 2\pi$, can be replaced by its average over one cycle of the piston of length 2π using the integral

$$\frac{1}{2\pi}\int_{\theta=\pi}^{2\pi} \frac{\sin\{\theta + \arcsin[\sin(\theta)/(L/R)]\}}{\cos\{\arcsin[\sin(\theta)/(L/R)]\}} d\theta = -\frac{1}{\pi}. \tag{18}$$

Carrying the average of Eq. (13) over one cycle, we find, approximately, using $F^\dagger = -1$ for $0 < \theta \leqslant 2\pi$,

$$\ddot{\bar{\theta}}^\dagger = T_m^\dagger \dot{\bar{\theta}}^\dagger \left[2 - R^\dagger \dot{\bar{\theta}}^\dagger\right] - \frac{\mathcal{L}^\dagger}{\pi}. \tag{19}$$

To solve this ODE, we first factorize its quadratic right-hand-side,

$$T_m^\dagger \dot{\bar{\theta}}^\dagger \left[2 - R^\dagger \dot{\bar{\theta}}^\dagger\right] - \frac{\mathcal{L}^\dagger}{\pi} = -R^\dagger T_m^\dagger (\dot{\bar{\theta}}^\dagger - \alpha_+)(\dot{\bar{\theta}}^\dagger - \alpha_-), \tag{20}$$

[8] Matlab constantly readjusts the integration time step to save CPU time.

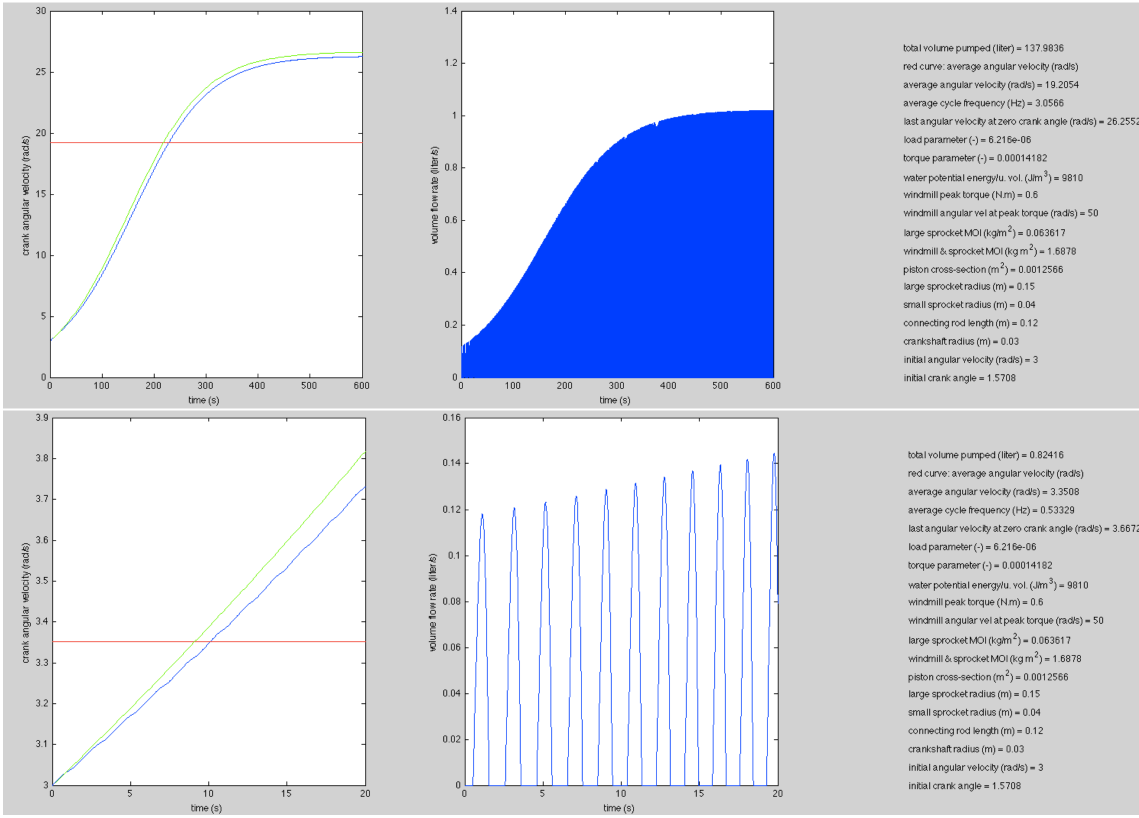

FIGURE 3. Example of MATLAB output. On the left graphs, the blue, green and red lines are, respectively, the computed crankshaft angular velocity (rad/s), the prediction of Eq. (24), and the computed average value. On the right graphs, the blue line of volume flow rate (liter/s). Note that volume flow rate is intermittent, as expected of this single-cylinder, single acting piston engine.

where

$$\alpha_{\pm} = \frac{\pi T_m^\dagger \pm \sqrt{\pi^2 T_m^{\dagger 2} - \pi R^\dagger T_m^\dagger \mathcal{L}^\dagger}}{\pi R^\dagger T_m^\dagger}. \tag{21}$$

Our ability to factorize is governed by the sign of the argument in the square root. It is positive if and only if

$$T_m^\dagger > R^\dagger \mathcal{L}^\dagger / \pi, \tag{22}$$

which sets the minimum windmill performance to handle the load. To integrate Eq. (19), we write it

$$-R^\dagger T_m^\dagger dt^\dagger = \frac{1}{(\alpha_+ - \alpha_-)}\Big[\int_{\alpha=\dot\theta_0}^{\dot\theta^\dagger} \frac{d\alpha}{(\alpha-\alpha_+)} - \int_{\alpha=\dot\theta_0}^{\dot\theta^\dagger} \frac{d\alpha}{(\alpha-\alpha_-)}\Big], \tag{23}$$

with the solution

$$\Big(\frac{\dot\theta^\dagger - \alpha_+}{\dot\theta^\dagger - \alpha_-}\Big)\Big(\frac{\alpha_0 - \alpha_-}{\alpha_0 - \alpha_+}\Big) = \exp\Big[-\frac{2}{\pi}t^\dagger\sqrt{\pi^2 T_m^{\dagger 2} - \pi R^\dagger T_m^\dagger \mathcal{L}^\dagger}\Big], \tag{24}$$

from which $\dot\theta^\dagger$ can be found in terms of the exponential. In Fig. 3, the green line is $\dot\theta^\dagger$; it agrees well with the Matlab numerical solution of the ODE.

Equation (24) indicates that the windpump reaches its steady operation on a dimensionless time scale

$$\tau^\dagger = \frac{\pi}{2\sqrt{\pi^2 T_m^{\dagger 2} - \pi R^\dagger T_m^\dagger \mathcal{L}^\dagger}}. \tag{25}$$

As we would expect, this time increases as the load grows (greater $\mathcal{L}^\dagger$), or as the windmill torque decreases (smaller $T_m^\dagger$). As Eq. (22) draws near equality, $\tau^\dagger \to \infty$ i.e., the windmill is overwhelmed by the load.