5th International Conference on Multiphase Flow, ICMF'04
Yokohama, Japan, May 30-June 4, 2004
Paper No.K13

Role of Couple Stresses in Shallow Granular Flows down a Bumpy Incline

Michel Y. Louge[1]

1: Sibley School of Mechanical and Aerospace Engineering, Cornell University, Ithaca, NY, USA, MYL3@cornell.edu

Abstract We extend the micropolar fluid theory of Hayakawa and Mitarai, et al. [PRL **88**, 174301 (2002)] to dense, relatively shallow flows of spherical grains down an inclined plane on which spherical bumps have been affixed. We update the model of Louge [PRE **67**, 061303 (2003)], and show the role played by couple stresses in establishing the solid volume fraction in the core of the flow.

1 Introduction

Steady, fully-developed (SFD) shallow flows of spherical grains down rigid bumpy inclined planes far away from the confinement of side walls are a convenient laboratory paradigm for more complex geophysical granular phenomena like rock slides. If the bump size is on the order of the grain diameter d, these flows are entirely mobilized and the depth-averaged velocity $\bar{u}$ grows with the $3/2$ power of the depth h. Pouliquen's experiments [1] showed that they only exist within a range of angles of inclination and at a minimum depth h_{stop} that decreases with the angle of inclination α. The numerical simulations of Silbert, *et al* [2] further revealed that the ratio $\bar{u}/h^{3/2}$ is independent of normal restitution e and interparticle friction μ, that the mean velocity vanishes at the base, that the granular temperature T grows with depth in the region away from the free surface and the bottom boundary, and that, remarkably, the solid volume fraction ν is independent of depth except within a grain diameter or so of the free surface.

Louge [3] proposed a theory to capture the observations of Pouliquen [1] and Silbert, *et al* [2]. To simplify the treatment of the governing equations, he distinguished three regions in the flow: a thin layer near the free surface where the grains interact only through collisions; a "core" where both impulsive and enduring particle contacts coexist; and a basal layer within a few grain diameters of the rigid bottom boundary where the angular momentum balance sets the relative magnitude of impulsive and enduring stresses. His theory predicted the range of inclination angles at which steady, fully-developed flows are observed, the corresponding shape of the mean and fluctuation velocity profiles, the dependence of the flow rate on inclination, flow height, interparticle friction and normal restitution coefficient, and the dependence of the height of basal flows on inclination. However, Louge [3] could not explain why ν decreases with α while remaining invariant in the depth. Our objective is to do so.

We begin with a summary of Louge's theory and discuss its closure. Focusing on the core, we examine alternative explanations for the invariance of volume fraction with depth, and discuss the paradoxes they raise. We then show how the invariance of couple stresses in the core implies the invariance of ν there, and how solutions of the governing equations in the basal layer yield the dependence of ν on inclination and its independence on other parameters.

2 Background

In SFD gravitational flow, the momentum balances along the flow and in its depth reduce to

$$\frac{dS}{dy} = -\rho_s \nu g \sin\alpha, \tag{1}$$

and

$$\frac{dN}{dy} = -\rho_s \nu g \cos\alpha, \tag{2}$$

where S and N are, respectively, the shear stress and normal stress on surfaces parallel to the base, ρ_s is the material density of the grains, g is the gravitational acceleration, and y is the coordinate perpendicular from the base and pointing toward the free surface. In dense flow, these equations yield, approximately,

$$S \approx \rho_s \bar{\nu} g \sin\alpha (h - y) \tag{3}$$

and

$$N \approx \rho_s \bar{\nu} g \cos\alpha (h - y), \tag{4}$$

in which the depth-averaged volume fraction $\overline{\nu}$ was substituted for its local value. It follows that the ratio of shear to normal stress is constant and equal to the tangent of the angle of inclination,

$$S/N = \tan\alpha. \tag{5}$$

Louge [3] followed Savage [4] and others in superposing two components of the stresses,

$$S = S_I + S_E \tag{6}$$

and

$$N = N_I + N_E, \tag{7}$$

where the subscript I refers to impulsive interactions leading to rate-dependent stresses and the subscript E denotes enduring contacts associated with rate-independent stresses. Louge modeled the latter with an internal friction μ_E such that

$$S_E/N_E = \mu_E. \tag{8}$$

For convenience, he also defined the fraction η of the total shear stress that is rate-independent,

$$\eta \equiv S_E/S. \tag{9}$$

For the rate-dependent stresses, he invoked the constitutive relations of Jenkins and Richman [5] for nearly elastic spheres,

$$S_I = A_1(\nu)\nu^2 g_{12}\rho_s d\sqrt{T}\frac{du}{dy}, \tag{10}$$

and

$$N_I = A_4(\nu)\nu^2 g_{12}\rho_s T, \tag{11}$$

where $A_1(\nu) = (8/5\sqrt{\pi})[1 + (\pi/12)(1 + 5/8\nu g_{12})^2]$, $A_4(\nu) = 4(1 + 1/4\nu g_{12})$ and g_{12} is the pair distribution of Carnahan and Starling [6] corrected by Torquato for high volume fractions [7]. To determine the depth profile of granular temperature, he wrote the energy balance

$$-\frac{dq}{dy} + (S_I + S_E)\frac{du}{dy} - \gamma = 0, \tag{12}$$

which involves the flux of fluctuation energy

$$q = -A_2(\nu)\nu^2 g_{12}\rho_s d\sqrt{T}\frac{dT}{dy}, \tag{13}$$

with $A_2 = (4/\sqrt{\pi})[1 + (9\pi/32)(1 + 5/12\nu g_{12})^2]$, and the volumetric rate of energy dissipation

$$\gamma = A_3 \nu^2 g_{12} \rho_s T^{3/2}/d, \tag{14}$$

where, in the limit of nearly elastic, nearly frictionless spheres, $A_3 = (24/\sqrt{\pi})(1 - e_{eff})$ and e_{eff} is an effective coefficient of restitution combining the collisional energy dissipation associated with impact inelasticity and friction [8].

Unfortunately, combining Eqs. (3) through (14) produces only three independent ordinary differential equations (ODE) representing balances for two components of the momentum and for the fluctuation energy. Thus, because the problem involves four dependent variables (u, T, ν and η), it is not closed.

To provide closure, Louge [3] assumed that ν is invariant with depth and known in terms of η through a empirical expression derived from the simulations of Silbert, *et al* [2]. Because Louge's predictions of $\bar{u}/h^{3/2}$ were relatively insensitive to the actual value of ν, his empirical closure was sufficient in practice, albeit disappointing.

Other theories circumvented the problem of closure differently. Johnson, Nott and Jackson [9] postulated an empirical expression for the dependence of the normal stress on volume fraction. Jenkins [10] and others ignored the presence of enduring contacts, $\eta = 0$. To recover qualitative features of the flow in this case, Bocquet, *et al* [11] adjusted the dependence of transport coefficients on volume fraction.

Our objective is to update Louge's theory by explaining the observed invariance of ν with y. We will frame the discussion in terms of alternative explanations, which, as we will show, only provide qualitative agreement with observations.

3 The Core

In the dense, relatively shallow flows of Pouliquen [1] and Silbert, *et al* [2], most of the depth belongs to the core. Consequently, the overall mass flow rate is dominated by the contribution of that region. Thus, to understand the behavior of the flow, it is instructive to focus on the core first. There, the flux gradient term in Eq. (12) is negligible, and the resulting energy balance yields

$$\frac{(1-\eta)\tan^2\alpha}{(1-\eta\tan\alpha/\mu_E)^2} = \frac{A_1 A_3}{A_4^2}, \tag{15}$$

Like Jenkins [10], Louge [3] noted that, because the flow is dense, the functions A_1, A_2, A_3 and A_4 tend to constants that are independent of ν. In this case, Eq. (15) is a quadratic possessing physical solutions for η iff the inclination lies within the range

$$\frac{\mu_E}{1 + A_4^2\mu_E^2/(4A_1A_3)} < \tan\alpha \leq \sqrt{A_1A_3}/A_4. \tag{16}$$

There, the core granular temperature and mean velocity profiles are, approximately,

$$\frac{T}{gd} \simeq \Xi \frac{A_4}{A_1A_3}\frac{\bar{\nu}\sin\alpha}{\nu^2 g_{12}}\Big(\frac{h-y}{d}\Big), \tag{17}$$

and

$$\frac{u}{\sqrt{gd}} \simeq \frac{2}{3}\frac{A_4^{3/2}}{A_1}\Xi^{3/2}\sqrt{\frac{\bar{\nu}\sin\alpha}{A_1A_3\nu^2 g_{12}}}\Big(\frac{h}{d}\Big)^{3/2}\Big\{1 - \Big(\frac{h-y}{h}\Big)^{3/2}\Big\}, \tag{18}$$

where

$$\Xi \equiv \frac{S_I}{N_I} = \sqrt{\frac{A_1A_3}{A_4^2}(1-\eta)}. \tag{19}$$

and η is the smallest solution of Eq. (15) [3].

The difficulty with Louge's approach is that the volume fraction remains undetermined [3]. An alternative is to impose closure by assuming that the flow only involves collisional interactions, $\eta \equiv 0$. However, such assumption leads to paradoxical results.

First, the dense collisional limit, for which A_1, A_2 and A_4 are independent of ν, makes the right-hand-side of Eq. (15) constant, and therefore only permits the last two terms in Eq. (12) to balance at the single inclination $\tan\alpha = \sqrt{A_1 A_3}/A_4$. Consequently, the flux gradient term in Eq.(12) cannot be neglected in general. Second, Jenkins [10] showed that dense SFD flows only exist at a fixed value of h, which contradicts observations of SFD flows with a wide range of depths.

To resolve this conundrum, one can keep the full dependence of A_1, A_2 and A_4 on ν. In this case, substituting $\eta \equiv 0$ in Eq. (12) leads to a quadratic equation with unknown variable $1/4\nu g_{12}$ and parameter $\psi \equiv 5\pi \tan\alpha^2/12(1 - e_{eff})$. Encouragingly, the solution for νg_{12} and, consequently, for ν are indeed independent of y. When $\psi \in [25\pi/3(16 + 3\pi), 1 + \pi/12[$, there are two solutions for $1/4\nu g_{12}$, but only one agrees with observations that ν decreases with steeper inclination. When $\psi \in [1 + \pi/12, 25\pi/48[$, there is only one solution, which is also consistent with the observed trend. Thus, if $\eta \equiv 0$, Eq. (15) admits physical solutions for ν such that

$$\frac{1}{4\nu g_{12}} = \frac{48\psi - 10\pi + \sqrt{\psi(4 + 3\pi/4) - 25\pi/12}}{25\pi - 48\psi}, \tag{20}$$

iff $\psi \in [25\pi/3(16 + 3\pi), 25\pi/48[$ or, equivalently, within the narrow range of inclinations

$$\sqrt{\frac{20}{16 + 3\pi}(1 - e_{eff})} \leq \tan\alpha \leq \sqrt{\frac{5}{4}(1 - e_{eff})}. \tag{21}$$

In that range, the product $\nu g_{12} \in]0, 5\pi/(64 - 8\pi)]$ is small enough to invoke the Carnahan-Starling pair distribution for g_{12} [6]. Unfortunately, this yields volume fractions $\nu \in]0, 0.36]$ that are unrealistic for dense flows. Consequently, the purely collisional constitutive relations of Jenkins and Richman [5] are not alone sufficient to capture the dense gravitational flows of interest.

Bocquet, *et al* [11] adopted instead other constitutive relations that diverge more rapidly with ν, and thus managed to capture qualitative features of the flows. However, because the relations of Jenkins and Richman [5] have succeeded in relatively dense microgravity experiments and in numerical simulations where enduring contacts are absent [12], it is not appropriate to dismiss them so casually. Instead, it is likely that the assumption of a purely collisional flow is inappropriate for dense flows down inclines.

In this context, we seek another explanation for the invariance of the volume fraction in the core.

4 Couple Stresses

Because contact forces are not exerted on the center of mass of individual grains, but rather on their external surfaces, it is possible for the mean spin of individual particles to differ from the rotation rate of the mean velocity field. In this case, the usual stress tensor is augmented by an asymmetric part that is proportional to this difference. For SFD flows, the mean angular momentum of the grains involves a balance between the gradient of couple stresses and the torque due to the asymmetric part of the stress tensor [13]. The stress tensor τ_{ij} and the couple stress tensor C_{ij} represent, respectively, the j-component of the surface force and surface torque acting on the plane of normal i per unit area.

Mitarai, *et al* [13] wrote the balances of linear and angular momentum for dilute SFD flows down an incline. Extending their results to dense flows with both impulsive and enduring contacts, we modify Eqs. (1) as

$$\frac{d}{dy}\left[S_I + S_E + \mu_R\Big(\frac{du}{dy} + 2\omega\Big)\right] = -\rho_s \nu g \sin\alpha, \tag{22}$$

where μ_R is the microrotation viscosity and ω is the mean granular spin. In SFD flows, the balance of angular momentum reduces to

$$\frac{dC_{yz}}{dy} = 2\mu_R\left(\frac{du}{dy} + 2\omega\right). \tag{23}$$

Although the mean grain spin exhibits spatial oscillations near boundaries that correspond to the counter-rotation of adjacent granular layers [14, 15], we adopt the simple constitutive relation for the couple stress that Mitarai, *et al* used for dilute flows,

$$C_{yz} = \mu_B\frac{d\omega}{dy}, \tag{24}$$

where μ_B is one of three coefficients of angular viscosity [13].

We assume that the translation and rotation temperatures are equal, and that the dissipation in Eq. (14) accounts for the dissipation of rotational fluctuation energy [8]. Then, we simply augment the fluctuation energy balance in Eq. (12) with a term that captures the production of fluctuation energy by the working of the couple stress through the gradient of the mean granular spin,

$$-\frac{dq}{dy} + S\frac{du}{dy} + C_{yz}\frac{d\omega}{dy} - \gamma = 0. \tag{25}$$

In the core, this additional term is much smaller than Sdu/dy. However, it is significant in the basal layer.

Lun [16] calculated μ_R for dense, inelastic, slightly rough spheres. Unfortunately, he assumed an impact model based upon a constant tangential restitution coefficient β, rather than the constant friction that is observed with actual spherical grains. To translate his expression to more common parameters, we calculate $\beta = \min(\beta_0, -1+(7/2)\mu(1+e)/\Psi_1)$, where β_0 is the coefficient of tangential restitution for rolling impacts, e is the coefficient of normal restitution, μ is the coefficient of friction, and Ψ_1 is the tangent of the incident impact angle [17], which we estimate as $\Psi_1 \simeq |\omega|d/\sqrt{T}$. With these assumptions, we find

$$\mu_R \simeq \frac{\nu^2 g_{12}}{\sqrt{\pi}}\sqrt{T}\min\left[\frac{\mu(1+e)}{|\omega|d/\sqrt{T}}, \frac{2}{7}(1+\beta_0)\right]. \tag{26}$$

In the absence of calculations for μ_B, we adopt

$$\mu_B \simeq \mu_R d^2. \tag{27}$$

In the core, we expect that the mean granular spin and the rotation rate of the mean velocity field are equal,

$$\omega = -\frac{1}{2}\frac{du}{dy}. \tag{28}$$

Then, Eq. (23) implies that the couple stress is invariant there. Combining Eqs. (24) and (28), it is

$$C_{yz_{core}} = -\frac{1}{2}\mu_B\frac{d^2u}{dy^2}. \tag{29}$$

This invariance agrees with the expected dependence of $\mu_B d^2u/dy^2$ with y; because $\mu_B \propto \sqrt{T} \propto \sqrt{h-y}$ and $du/dy \propto \sqrt{h-y}$, the product $\mu_B d^2u/dy^2$ is indeed invariant in the depth. Combining Eqs. (17), (18), (19) and (27) then yields, in the dense limit,

$$C_{yz_{core}} = \rho_s g d^2 \sin\alpha\frac{\nu}{4A_1\sqrt{\pi}}(1-\eta)\min\left[\frac{A_1}{2\Xi}\mu(1+e), \frac{2}{7}(1+\beta_0)\right]. \tag{30}$$

To infer the core volume fraction from this expression, we must evaluate $C_{yz_{core}}$. We do so by writing a system of ODEs for u, ω, T, q and C_{yz} in the base layer, from which we extract $C_{yz} = C_{yz_{core}}$ at the location $y = b$ just beneath the core.

First, we combine Eqs. (2), (22) and (23), integrate the result between $y = b$ down to any $y \in [0, b]$ and, because $S = N \tan\alpha$ and $dC_{yz}/dy = 0$ at $y = b$, we find

$$S + \frac{1}{2}\frac{dC_{yz}}{dy} = N \tan\alpha. \tag{31}$$

This result indicates that the couple stress gradient reduces the shear stress necessary to balance the streamwise component of gravity.

Next, from Eqs. (6), (7), (10), (23) and (31), we extract the two ODEs

$$\frac{du}{dy} = \left(\frac{1}{\mu_I + \mu_R}\right)[N(\tan\alpha - \mu_E) + \mu_E N_I - 2\mu_R\omega] \tag{32}$$

and

$$\frac{dC_{yz}}{dy} = 2\left(\frac{\mu_R}{\mu_I + \mu_R}\right)[N(\tan\alpha - \mu_E) + \mu_E N_I + 2\mu_I\omega], \tag{33}$$

where $\mu_I \equiv A_1(\nu)\nu^2 g_{12}\rho_s d\sqrt{T}$ is the granular shear viscosity, and the magnitudes of N and N_I are given by Eqs. (4) and (11).

In principle, if ν is allowed to vary in the basal layer, one equation is still lacking to close the problem in that region. However, the simulations of Silbert, *et al* [2] indicate that, apart from inevitable spatial oscillations from the ordering near the wall, ν is nearly invariant in the basal layer as well, except at inclinations that are close to the maximum angle for SFD flow. In two-dimensional flow, ν is invariant all the way to the base at any angle [2]. Thus, rather than writing another governing equation, we assume that the value of ν in the core persists in the basal layer.

With this assumption, we solve the coupled set of ODEs (13), (24), (25), (32), and (33) by shooting downward from $y = b$, where $u = u_b$, and q, T, ω and C_{yz} are matched to their respective values calculated in the core from Eqs. (13), (17), (28), and (30). We fix the velocity u_b by setting $u = 0$ where the base is reached. We find the thickness b of the base layer when the integration satisfies the flux boundary condition at the bumpy base. There, because there is no relative velocity between the base and the flowing grains, the fluctuation energy flux reduces to

$$q = -D, \tag{34}$$

where D is a rate of fluctuation energy dissipation per unit surface of the base. For purely collisional flows, Jenkins and Richman [18] calculated

$$D = 2\sqrt{\frac{2}{\pi}}(1 - e_w)\frac{N_I\sqrt{T}}{1 + \cos\theta}, \tag{35}$$

where $\sin\theta \equiv (d - d_b + \Delta)/(d + d_b)$ is a measure of the bumpiness of the boundary with spherical bumps of diameter d_b, Δ is the mean separation between the centers of two adjacent bumps, and e_w is the coefficient of normal restitution in the impact of a flow sphere with a bump. As Louge [3] noted, because grains experience enduring contacts with the base, Eq. (35) probably under-predicts the surface dissipation. Combining Eq. (13), (34) and (35), we find the boundary condition

$$\frac{d\sqrt{T}}{dy} = b_1\frac{\sqrt{T}}{d}, \tag{36}$$

where

$$b_1 = \frac{4}{a_2}\sqrt{\frac{2}{\pi}}\left(\frac{1-e_w}{1+\cos\theta}\right). \tag{37}$$

It remains to determine the solid volume fraction. We do so by applying the boundary condition for couple stress at the base. By shrinking the thickness of a pillbox control volume, Jenkins [19] established that the couple stress on the base is equal to the rate of supply of angular momentum per unit are of the wall, or

$$C_{yz} = L, \tag{38}$$

where L is the product of the rate of collisions per unit surface of the wall and the angular momentum lost in a collision,

$$L \sim A_L\Big[\nu g_0 \frac{\sqrt{T}}{d^3}\Big]\Big(\rho_s d^4\sqrt{T}\min[\mu_w(1+e_w),(2/7)(1+\beta_{0_w})]\Big). \tag{39}$$

In this expression, $g_0(\nu)$ is the pair distribution function of spheres and wall bumps, μ_w and β_0 are, respectively, the friction and tangential restitution for their impacts, and A_L is a constant of order one. Because wall collisions involve the contacts of two spheres, we invoke for g_0 a dense correction to the Carnahan and Starling pair distribution [6] similar to Torquato's [7]. However, we expect the wall-induced layering to make g_0 rise faster with ν than g_{12}. Thus, we adopt

$$g_0(\nu) = \frac{2-\nu}{2(1-\nu)^3} \text{ for } 0 \leq \nu \leq \nu_f \tag{40}$$

and

$$g_0(\nu) = \left(\frac{(2-\nu_f)}{2(1-\nu_f)^3}\right)\left(\frac{(\nu_c-\nu_f)}{(\nu_c-\nu)}\right)^{1.3} \text{ for } \nu_f < \nu < \nu_c, \tag{41}$$

where $\nu_f = 0.49$ and $\nu_c = 0.64$.

To find the solid volume fraction, we guess a value, solve the set of ODEs as outlined above, compare the resulting couple stress at the base with L in Eq. (39), and iterate ν until the boundary condition (38) is satisfied.

Figure 1 shows typical profiles through the depth. It confirms that the energy flux is small in the core. Consistent with the observations of Silbert, *et al* [2], the temperature reaches a maximum in the basal layer. Unlike what Louge assumed [3], this maximum does not occur at the edge of the core. The couple stress is smaller in the core than it is in the base layer.

Figure 2 compares our prediction for the dependence of ν on angle of inclination with the observations of Silbert, *et al* [2]. Other calculations, which are not reported here, indicate that the core solid volume fraction is nearly independent of the overall depth and of the interparticle friction μ, once again in agreement with the simulations of Silbert, *et al*.

5 Acknowledgments

The author gratefully acknowledges the advice and guidance of Namiko Mitarai and Haitao Xu. He is also endebted to Alexandre Valance, Renaud Delannay, Patrick Richard and Nicolas Taberlet for illuminating discussions on inclined flows. This work was supported by the National Science Foundation under a US-France Cooperative Research grant INT-0233212 and by NASA under contracts NAG3-2705 and NCC3-797.

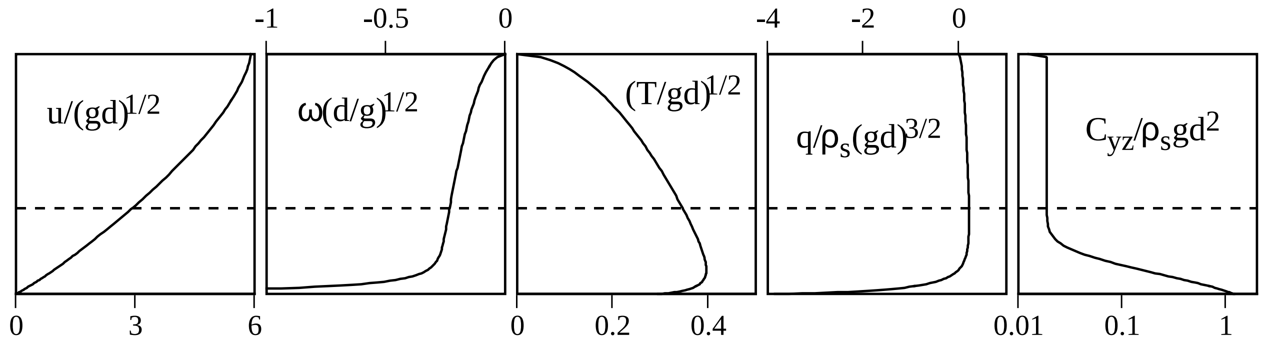

Figure 1: Profiles of dimensionless mean velocity, angular velocity, fluctuation velocity, energy flux and couple stress for conditions of Pouliquen's "system 1" [1] with $e_{eff} = 0.67$, $\mu_E = 0.42$, $\alpha = 25°$. We adopt $A_1 = 1.5$, $A_2 = 8.5$, $\mu = \mu_w = 0.525$, $b_1 = 1.5$ and $A_L = 1.8$. We calculate $\nu = 0.59$. The ordinate represents the upward coordinate normal to the base. The dashed line indicates the boundary between the core and the basal layer.

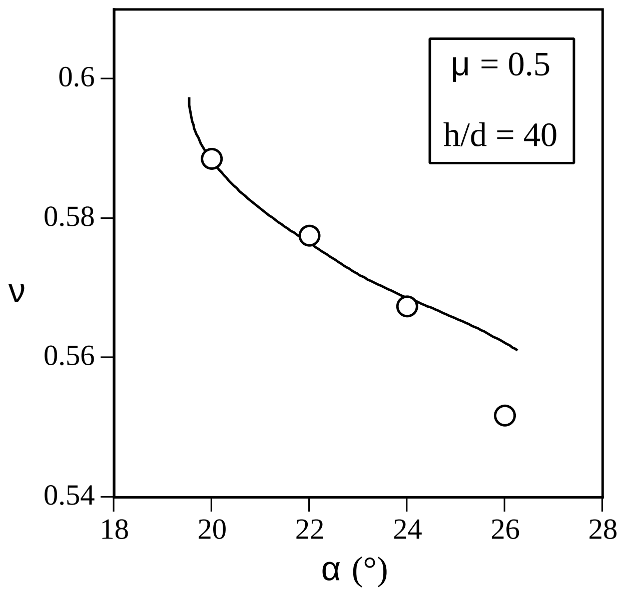

Figure 2: Predictions of solid volume fraction versus angle of inclination for the overall depth and friction shown. The symbols are data from system L3 of Silbert, *et al* and the solid line represent our predictions with $e_{eff} = 0.83$ and $\mu_E = 0.40$. We adopt $A_1 = 2$, $A_2 = 8.5$, $\mu = \mu_w = 0.5$, $b_1 = 1.5$ and $A_L = 2.4$.

References

[1] O. Pouliquen, Scaling laws in granular flows down rough inclined planes, Phys. Fluids 11 (1999) 542–548.

[2] L. Silbert, D. Ertaş, G. Grest, T. Halsey, D. Levine, S. Plimpton, Granular flow down an inclined plane: Bagnold scaling and rheology, Phys. Rev. E 64 (2001) 051302.

[3] M. Louge, Model for dense granular flows down bumpy inclines, Phys. Rev. E 67 (2003) 061303–1:10.

[4] S. Savage, Gravity flow of cohesionless granular materials in chutes and channels, J. Fluid Mech. 92 (1979) 53–96.

[5] J. T. Jenkins, M. W. Richman, Grad's 13-moment system for a dense gas of inelastic spheres, Arch. Rat. Mech. Anal. 87 (1985) 355–377.

[6] N. Carnahan, K. Starling, Equations of state for non-attracting rigid spheres, J. Chem. Phys. 51 (1969) 635–636.

[7] S. Torquato, Nearest-neighbor statistics for packings of hard spheres and disks, Phys. Rev. E 51 (1995) 3170–3182.

[8] J. Jenkins, C. Zhang, Kinetic theory for identical, frictional, nearly elastic spheres, Phys. Fluids 14 (2002) 1228–1235.

[9] P. Johnson, P. Nott, R. Jackson, Frictional-collisional equations of motion for particulate flows and their application to chutes, J. Fluid Mech. 210 (1990) 501–535.

[10] J. Jenkins, Hydraulic theory for a debris flow supported on a collisional shear layer, IAHR, Tokyo, 1993, pp. 1–12.

[11] L. Bocquet, J. Errami, T. Lubensky, Hydrodynamic model for a dynamical jammed-to-flowing transition in gravity driven granular media, Phys. Rev. Lett. 89 (2002) 184301–1:4.

[12] H. Xu, M. Louge, A. Reeves, Solutions of the kinetic theory for bounded collisional granular flows, Continuum Mech. Thermodyn. 15 (2003) 321–349.

[13] N. Mitarai, H. Hayakawa, H. Nakanishi, Collisional granular flow as a micropolar fluid, Phys. Rev. Lett. 88 (2002) 174301–1:4.

[14] Y. Khidas, Etude expérimentale du frottement et des rotations dans des milieux granulaires modèles, Thèse de Doctorat, Université de Rennes I, 2001.

[15] M. Louge, Computer simulations of rapid granular flows of spheres interacting with a flat, frictional boundary, Phys. Fluids 6 (1994) 2253–2269.

[16] C. Lun, Kinetic theory for granular flow of dense, slightly inelastic, slightly rough spheres, J. Fluid Mech. 233 (1991) 539–559.

[17] S. Foerster, M. Louge, H. Chang, K. Allia, Measurements of the collision properties of small spheres, Phys. Fluids 6 (1994) 1108–1115.

[18] J. Jenkins, M. Richman, Boundary conditions for plane flows of smooth, nearly elastic, circular disks, J. Fluid Mech. 171 (1986) 53–69.

[19] J. Jenkins, Boundary conditions for rapid granular flows: Flat, frictional walls, J. Applied Mech. 59 (1992) 120–127.