The role of van der Waals adhesion for spheres escaping from an impact

Michel Louge

March 7, 2010

This brief article introduces the concept of a minimum escape speed necessary for spheres to recoil from an impact in which van der Waals adhesion is important, but gravity is not. It calculates trajectories in case spheres escape or not.

1 Escape speed

Consider two spheres of radius R, mass m made of a material of Hamaker constant A. In this one-dimensional problem, we set the origin at the contact point and define the positive x-axis away from the impact. Van der Waals attraction is the only force in this problem,

$$F_v = -\frac{AR}{12H^2}, \tag{1}$$

where the distance H is shown in Fig. 1. Using $y \equiv x - R + H_0/2$, the equation of motion of one of the spheres is

$$m\frac{d^2y}{dt^2} = -\frac{AR}{48y^2} \equiv \frac{dV}{dy}, \tag{2}$$

where we define the potential energy $V = (AR)/(48y)$.

Without loss of generality, we assume that the spheres are recoiling from impact with an initial velocity $v_0 > 0$ at the instant of separation $t = 0$. By analogy with planetary gravitation, we equate the kinetic and potential energies at recoil to calculate the escape speed

$$v_e = \sqrt{\frac{AR}{12mH_0}}, \tag{3}$$

such that any sphere with $v_0 > v_e$ will never return to impact again. We made time and distance dimensionless with $\tau = R\sqrt{48m/A}$ and R, respectively, and denote dimensionless quantities with a dagger †. The equation of motion becomes

$$\frac{d^2y^\dagger}{dt^{\dagger 2}} = -\frac{1}{y^{\dagger 2}}, \tag{4}$$

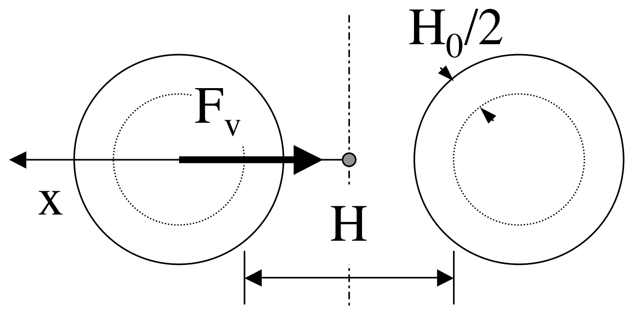

Figure 1: Sketch of two spheres recoiling from an impact at the contact point marked by a gray dot. Each is subject to an attractive van der Waals force, shown on the left sphere. The distance H_0 models atomic separation at contact.

subject to $y^\dagger = \epsilon$ and $dy^\dagger/dt^\dagger = v_0^\dagger \equiv v_0\tau/R$ at $t^\dagger = 0$, where $\epsilon \equiv H_0/(2R)$. (Note that $v_e^\dagger = \sqrt{2/\epsilon}$). Equivalently, this differential equation can be written

$$\frac{dy^\dagger}{dt^\dagger} = \pm\sqrt{\frac{2}{y^\dagger} + v_0^{\dagger 2} - v_e^{\dagger 2}} = \pm\sqrt{2\left(\frac{1}{y^\dagger} - \frac{1}{\epsilon}\right) + v_0^{\dagger 2}}, \tag{5}$$

subject to $y^\dagger = \epsilon$ at $t^\dagger = 0$. Consider first the separation from impact ($v_0^\dagger > 0$), for which the sign in Eq. (5) is positive.

We define $a \equiv v_0^{\dagger 2} - v_e^{\dagger 2}$. If $a > 0$, the solution is

$$t^\dagger = \frac{\sqrt{ay^\dagger(2 + ay^\dagger)} - 2\text{arcsinh}[\sqrt{ay^\dagger/2}]}{a^{3/2}}, \tag{6}$$

whereby $y^\dagger$ grows monotonically with $t^\dagger$ *ad infinitum*, and the spheres escape from one another.

If $a < 0$, solutions are only possible for $\epsilon \leqslant y^\dagger \leqslant -2/a$. In the time interval $0 \leqslant t^\dagger \leqslant \pi/(-a)^{3/2}$, the spheres separate with a trajectory satisfying

$$t^\dagger = \frac{2\,\text{arcsin}[\sqrt{-ay^\dagger/2}] - \sqrt{-ay^\dagger(2 + ay^\dagger)}}{(-a)^{3/2}} \equiv f(y^\dagger, a). \tag{7}$$

After reaching maximum separation $y^\dagger = -2/a$ at $t^\dagger = \pi/(-a)^{3/2}$, they return toward one another in a time-symmetrical trajectory for which the sign in Eq. (5) is negative.

$$t^\dagger = \frac{2\pi}{(-a)^{3/2}} - f(y^\dagger, a). \tag{8}$$

until they hit again at $t^\dagger = 2\pi/(-a)^{3/2}$.

If $a \gtrsim 0$, the espace trajectory is found by a series expansion of Eq. (6)

$$t^\dagger = 2^{1/2}y^{\dagger 3/2}/3 - ay^{\dagger 5/2}/10\sqrt{2} + o(a^2). \tag{9}$$