Converting pressure to contact displacements in arbitrary axi-symmetric loading

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November 25, 2006

This brief article summarizes the integral relations between normal stress and displacement for arbitrary axi-symmetric loading of an elastic solid. References: K.L. Johnson: Contact Mechanics, Cambridge University Press (1985); G. Haiat, M.C. Phan Huy and E. Barthel: "The adhesive contact of viscoelastic spheres", J. of the Mech. and Phys. of Solids **51**, 69-99 (2003).

1 Calculating pressure from prescribed displacements

Haiat, et al provide the following relations for the radial dependence of normal stress $p(r)$ that results from an arbitrary displacement function $u(r)$

$$p(r) = \frac{2}{\pi}\int_r^a \frac{g'(s)}{\sqrt{s^2 - r^2}} ds, \tag{1}$$

where

$$g(r) \equiv \frac{E}{2(1-\nu^2)}\Big[u(r=0) + r\int_0^r \frac{u'(s)}{\sqrt{r^2 - s^2}} ds\Big]. \tag{2}$$

In these equations, $u(r)$ is the displacement of the contact surface from its unloaded position, a is the radius of the contact patch beyond which pressure vanishes, E is Young's modulus, ν is Poisson's ratio, and primes denote first derivatives.

We check this formulation with the Hertz problem. With the displacement

$$u(r) = \Big(\frac{1-\nu^2}{E}\Big)\frac{\pi p_0}{4a}(2a^2 - r^2) = \frac{\pi D p_0 a}{6}\Big[2 - (\frac{r}{a})^2\Big], \tag{3}$$

we find

$$u'(r) = -\frac{\pi}{3}\frac{D p_0}{a} r, \tag{4}$$

where $D \equiv (3/2)(1-\nu^2)/E$. Because $\int_0^r s ds/\sqrt{r^2 - s^2} = r$, we find

$$g(r) = \frac{\pi p_0 a}{4}[1 - (\frac{r}{a})^2]. \tag{5}$$

Plugging $g'(r) = -(\pi/2)p_0(r/a)$ into equation (1), we find

$$p(r) = -p_0[1 - (\frac{r}{a})^2]^{1/2}, \tag{6}$$

which is the familiar Hertz pressure.

Mathematica sometimes fails to integrate equation (2). This can be detected by comparing results of a formal integration and its numerical counterpart. In this case, it may be more advantageous to use an alternative form of equation (2),

$$g(r) = \frac{3}{4D}\frac{d}{dr}\int_0^r \frac{su(s)}{\sqrt{r^2 - s^2}}ds. \tag{7}$$

For the Hertzian problem, this alternate form produces the same $g(r)$ found in equation (5).

2 Calculating displacements from a prescribed pressure distribution

Conversely, Haiat, et al show that

$$u(r) = \frac{2}{\pi}\int_0^r \frac{\theta(s)}{\sqrt{r^2 - s^2}}ds, \tag{8}$$

where

$$\theta(r) \equiv -\frac{4}{3}D\int_r^a \frac{sp(s)}{\sqrt{s^2 - r^2}}ds. \tag{9}$$

Once again, we verify this relation with Hertz's solution. Using the pressure distribution of equation (6) in equation (9), we find

$$\theta(r) = \frac{\pi D p_0 a}{3}\left[1 - (\frac{r}{a})^2\right], \tag{10}$$

and, because $\int_0^r ds(1 - s^2/a^2)/\sqrt{r^2 - s^2} = (\pi/4)(2 - r^2/a^2)$, we recover the Hertz displacement in equation (3).

We can also retrieve the displacement for constant pressure $-p_v$. In this case,

$$\theta(r) = \frac{4Dp_v}{3}\sqrt{a^2 - r^2}, \tag{11}$$

and

$$u(r) = \frac{8aDp_v}{3\pi}\text{EllipticE}[(\frac{r}{a})^2]. \tag{12}$$

Note that Johnson calls the elliptic function $\mathbf{E}(r/a)$. In Mathematica, the argument is $(r/a)^2$!