Optical measurements of solid volume fraction (supercedes the April 18, 2004 version)

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This paper reports approximate predictions for the number of spheres that we expect to detect by shining a sheet of light through the window of the SiGMA cell, and by observing its intersection with the shiny metal spheres within.

1 Sphere Population

We distinguish the "bulk" solid volume fraction ν^* from its local "center-average" counterpart ν [1]. For simplicity, we assume that ν^* is a constant in the cell's cross-section. Consistent with the formalism of the kinetic theory, the center-average volume fraction is related to the number density n of spheres of diameter σ having their centers at the location of interest,

$$\nu = \frac{\pi}{6}\sigma^3 n \tag{1}$$

In this view, a sphere contributes its entire volume to the infinitesimal region in which its center resides.

Because the window orders spheres in layers, the center-average solid volume fraction exhibits spatial oscillations, which are described by a spatial (or "pair") distribution function such that

$$\nu = \nu^* g_{12}(\zeta; \nu^*), \tag{2}$$

where

$$\zeta \equiv \frac{y}{\sigma} - \frac{1}{2} \tag{3}$$

and y is the coordinate normal to and originating at the window.

The theory of Henderson, Abraham and Barker (HAB) [2] predicts the form of g_{12},

$$g_{12}(\zeta; \nu^*) = \frac{1 + 2\nu^*}{(1 - \nu^*)^2} - \frac{3}{5}\nu^* \frac{10 - 2\nu^* + \nu^{*2}}{1 + 2\nu^*}\alpha_1 \zeta + \frac{1}{5}\nu^*(3\theta_1 + 2\theta_2) \tag{4}$$

$$+12\nu^* \int_0^\zeta t[\alpha_1(t - \zeta) + \theta_1 + \theta_2]h_{11}(t)dt \tag{5}$$

$$+12\nu^* \int_\zeta^{1+\zeta} t(\zeta + 1 - t)^3[\theta_1 + \theta_2(\zeta + 1 - t)h_{11}(t)dt, \tag{6}$$

where $h_{11}(t) \equiv g_{11}(t) - 1$, $\alpha_1 \equiv (1 + 2\nu^*)^2/(1 - \nu^*)^4$, $\theta_1 \equiv -2\nu^*(1 + \nu^*/2)(1 + 2\nu^*)/(1 - \nu^*)^4$, and $\theta_2 \equiv \nu^*\alpha_1/2$. The function $g_{11}(t)$ is the Percus-Yevick radial distribution function for a hard sphere, where $1 \leqslant t < +\infty$ is the relative distance between the sphere centers [3]. In the Appendix, we reproduce a Mathematica macro that calculates the pair distribution, which we call `HAB[e_,x_]`, where `e_` represents the volume fraction and `x_` represents ζ. To speed up subsequent integrations, we produce a look-up table of values of `HAB[e_,x_]`, which we fit using the `Interpolation` command in Mathematica with the function `ITT[e_,x_]` (see Appendix).

The HAB theory allows us to calculate the number N of spheres with centers located within a relative distance ζ_1 from a window of area A,

$$\frac{N\sigma^2}{A} = \frac{6}{\pi}\nu^* \int_{\zeta=0}^{\zeta_1} g_{12}(\zeta;\nu^*)d\zeta. \tag{7}$$

The corresponding function is `HAINT[e_, z_]`, where `z_` represents ζ_1. It is illustrated in Fig. 1. A good approximation of `HAINT` is the function `HAINTAPPROX`.

The "volume-average" solid volume fraction $< \nu >$ is more intuitive than its center-average counterpart [1]. It is defined as the fraction of the volume that is occupied by solids. In this case, a sphere contributes volume to any infinitesimal region in which any of its points resides. The volume-average solid volume fraction is related to the HAB theory by an integral that "spreads" volume about the sphere center,

$$< \nu(\frac{y}{\sigma}) >= 6\nu^* \int_{-1/2}^{+1/2} \Big(\frac{1}{4} - t^2\Big) g_{12}(t - \frac{1}{2} + \frac{y}{\sigma};\nu^*)dt, \tag{8}$$

where g_{12} is given by Eq. (4). In the Mathematica macro, we denote $< \nu >$ by the symbol `NUVOL[e_, y_]`, where `y_` represents y/σ. To speed up the calculation of subsequent integrals involving $< \nu >$, the Mathematica macro creates a look-up table of `NUVOL` and generates by interpolation an approximate function of $< \nu >$ called `NUIT`. Figure 2 compares the form of $< \nu >$ and $\nu^* g_{12}$ near a flat wall. Note that oscillations are present in both, but that $< \nu >$ is always less than unity, while $\nu^* g_{12} > 1$ is possible. (For example, at $\nu^* = 0.4$, $\nu^* g_{12} = 2$).

2 Optical Extinction

A difficulty with the optical method is that spheres are opaque. This leads to occlusion of the laser sheet and extinction of the light returning to the camera. In general, it is difficult to predict how many spheres will not be detected because of occlusion effects.

As Lischer and Louge [4] showed, Monte-Carlo simulations of the geometrical optics can provide such predictions. In this paper, we calculate instead a simple estimate based on the Beer-Lambert extinction law [5]. For a random assembly of spheres, the extinction of a plane wave front of light of intensity I is given by

$$\frac{dI}{dy} = -\frac{3}{2}\kappa < \nu > \frac{I}{\sigma}, \tag{9}$$

where $\kappa \simeq 2$ [5]. For simplicity, we assume that the path of the incoming laser sheet penetrates the window at an angle α from the normal, and that the return path to the camera is perpendicular to the window (Fig. 3). To be detected, a sphere with center located at a distance y must be reached by the laser sheet traveling through an incoming extinction path of length

$$\frac{1}{\cos\alpha}\Big[y - \frac{\sigma}{2}\cos\Big(\frac{\alpha}{2}\Big)\Big], \tag{10}$$

followed by a reflection traveling through a vertical path of length

$$y - \frac{\sigma}{2}\cos\Big(\frac{\alpha}{2}\Big). \tag{11}$$

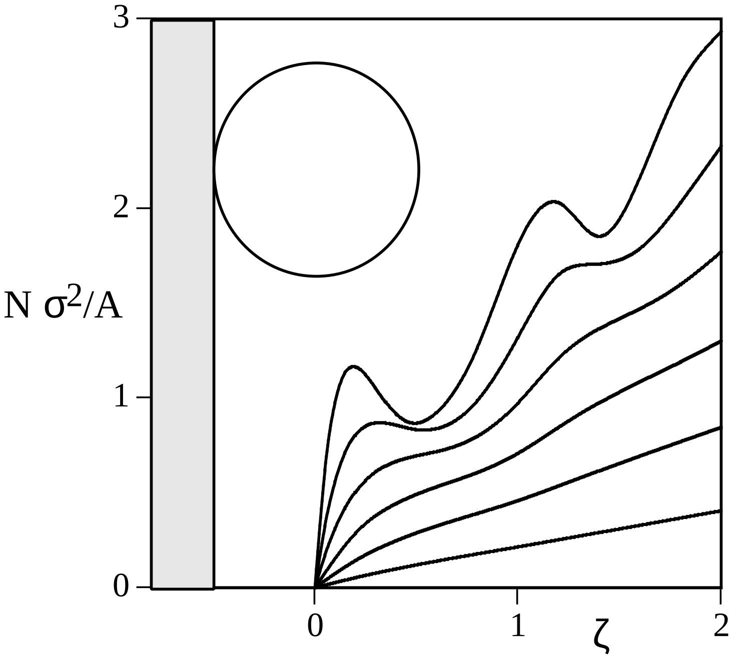

Figure 1: Number of spheres with centers located within the relative distance ζ from the window, made relative to the sphere diameter and the area of the window. The sketch also shows the window and a typical sphere. Curves from bottom to top: $\nu^* = 0.1$, 0.2, 0.3, 0.4, 0.5 and 0.6.

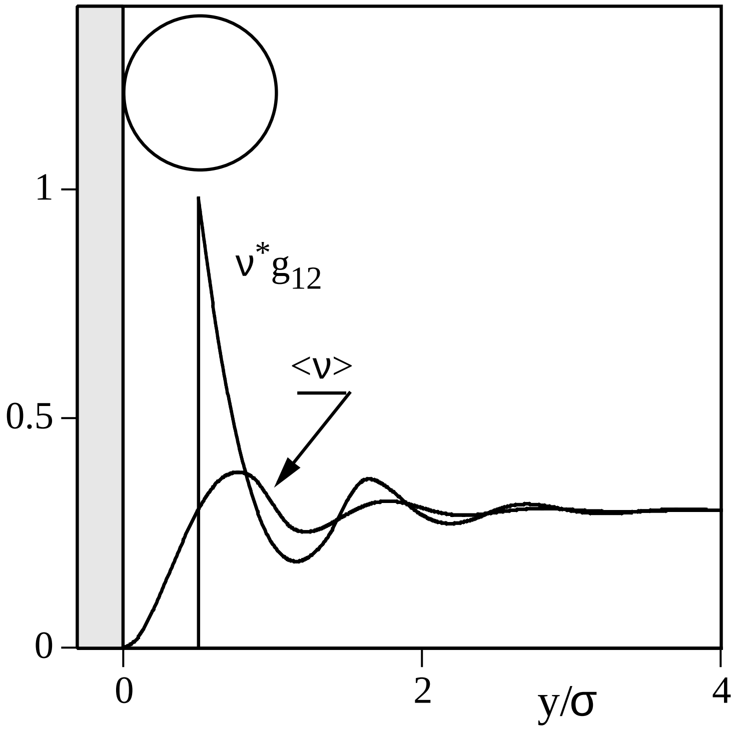

Figure 2: Profiles of "center-average" volume fraction $\nu^* g_{12}$ (Eq. 2) and "volume-average" volume fraction $< \nu >$ (Eq. 8) for $\nu^* = 0.3$. Note that $< \nu >$ peaks at $y > (1/2)\sigma$. The sketch also shows the window and a typical sphere.

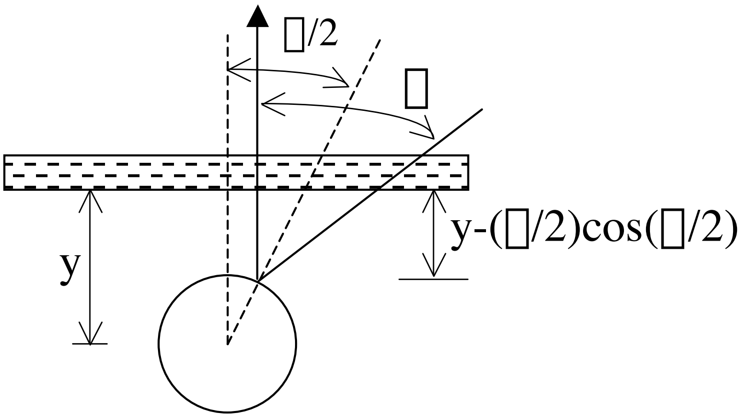

Figure 3: Optical path emerging from the window, striking the specular sphere, and returning perpendicular to the window.

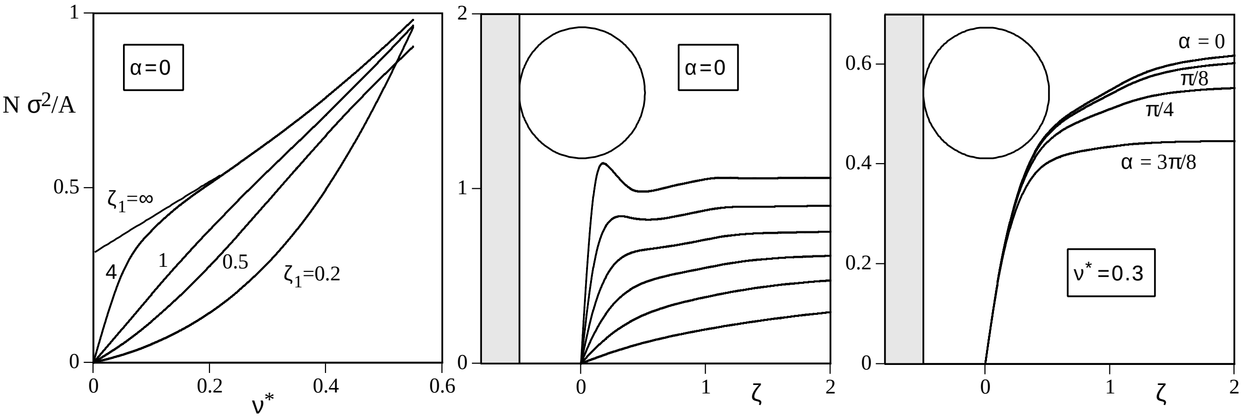

Figure 4: Number of detectable spheres with centers located within the range $\zeta = \left(\frac{y}{\sigma} - \frac{1}{2}\right) \in \left[0, \zeta_1\right]$, made relative to the sphere diameter and the area of the window, *versus* bulk solid volume fraction ν^* for the values of ζ_1 shown and $\alpha = 0$ (left); *versus* ζ_1 for $\alpha = 0$ and, from bottom to top, $\nu^* =$ 0.1, 0.2, 0.3, 0.4, 0.5 and 0.6 (center); *versus* ζ_1 at $\nu^* = 0.3$ for the values of α shown (right).

Thus, the total number of detectable spheres per dimensionless area is

$$\frac{N\sigma^2}{A}(\zeta_1;\nu^*) = \frac{6}{\pi}\nu^* \int_0^{\zeta_1} g_{12}(t;\nu^*) \exp\left[-\frac{3}{2}\kappa\Big(1+\frac{1}{\cos\alpha}\Big) \int_0^{t+\frac{1}{2}[1-\cos(\alpha/2)]} < \nu^*(y^*;\nu^*) > dy^* \right] dt \tag{12}$$

We calculate the corresponding integral `HABNUAPPROX[e_, al_, z_]` for $N\sigma^2/A$ in the Mathematica macro, where `e_` represent ν^*, `al_` represents α and `z_` represents ζ_1. Because the nested integrals take an increasingly long time to evaluate as ζ_1 grows, the function `HABNUAPPROX` uses the interpolations `ITT` and `NUIT`. In Fig. 4, the curve with $\zeta_1 = 4$ is nearly identical to that with with $\zeta_1 \to \infty$.

As Fig. 4 shows, the number of detectable spheres depends on the depth of the region $\zeta \in \left[0, \zeta_1\right]$ on which the optical method is trained. The optimum linearity of $N\sigma^2/A$ *versus* ν^* is obtained with $\zeta_1 = 1$. The angle of incidence α seems to play a relatively minor role.

3 Conclusions

From the simplified calculations outlined above, we have three recommendations:

[1] Train the optical system to detect spheres with centers located down to $\zeta_1 \simeq 1$, or equivalently at a distance $y \in \Big[0, 1.5\Big]\sigma$.

[2] Because curves in Fig. 4 are sensitive to ζ_1, carefully exclude spheres outside the region of interest.

[3] Calibrate the system to find the actual function $N\sigma^2/A$ *versus* ν^*.

4 Acknowledgments

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5 Appendix - Mathematica Macro

```
F[e_,0]=-(-6 + 6/(1 - 2*e + e^2) - (6*e)/(1 - 2*e + e^2))/3 -
         (2^(2/3)*(-1 + e)*e^2)/((1 - 2*e + e^2)*
         (-3*e - 3*e^2 + e^3 +3^(1/2)*(3*e^2 + 6*e^3 +
         e^4 - 2*e^5 + e^6)^(1/2))^(1/3)) +(2^(1/3)*(-3*e
         - 3*e^2 + e^3 +3^(1/2)*(3*e^2 + 6*e^3 + e^4 -
         2*e^5 + e^6)^(1/2))^(1/3))/(-1 + e) ;

F[e_,1]=-(-6 + 6/(1 - 2*e + e^2) - (6*e)/(1 - 2*e + e^2))/3 -
(-((2^(2/3)*(-1 + e)*e^2)/
((1 - 2*e + e^2)*(-3*e - 3*e^2 + e^3 +
     3^(1/2)*(3*e^2 + 6*e^3 + e^4 - 2*e^5 + e^6)^(1/2))^(1/3))) \
    + (2^(1/3)*(-3*e - 3*e^2 + e^3 +
   3^(1/2)*(3*e^2 + 6*e^3 + e^4 - 2*e^5 + e^6)^(1/2))^(1/3))/
    (-1 + e))/2 + I/2*3^(1/2)*
 ((2^(2/3)*(-1 + e)*e^2)/
    ((1 - 2*e + e^2)*(-3*e - 3*e^2 + e^3 +
   3^(1/2)*(3*e^2 + 6*e^3 + e^4 - 2*e^5 + e^6)^(1/2))^(1/3)) +
   (2^(1/3)*(-3*e - 3*e^2 + e^3 +
   3^(1/2)*(3*e^2 + 6*e^3 + e^4 - 2*e^5 + e^6)^(1/2))^(1/3))/
    (-1 + e)) ;

F[e_,2]=-(-6 + 6/(1 - 2*e + e^2) - (6*e)/(1 - 2*e + e^2))/3 -
(-((2^(2/3)*(-1 + e)*e^2)/
((1 - 2*e + e^2)*(-3*e - 3*e^2 + e^3 +
     3^(1/2)*(3*e^2 + 6*e^3 + e^4 - 2*e^5 + e^6)^(1/2))^(1/3))) \
    + (2^(1/3)*(-3*e - 3*e^2 + e^3 +
   3^(1/2)*(3*e^2 + 6*e^3 + e^4 - 2*e^5 + e^6)^(1/2))^(1/3))/
    (-1 + e))/2 - I/2*3^(1/2)*
 ((2^(2/3)*(-1 + e)*e^2)/
```

```
    ((1 - 2*e + e^2)*(-3*e - 3*e^2 + e^3 +
   3^(1/2)*(3*e^2 + 6*e^3 + e^4 - 2*e^5 + e^6)^(1/2))^(1/3)) +
   (2^(1/3)*(-3*e - 3*e^2 + e^3 +
   3^(1/2)*(3*e^2 + 6*e^3 + e^4 - 2*e^5 + e^6)^(1/2))^(1/3))/
    (-1 + e)) ;

L[t_,e_]=(1+e/2) t + 1 + 2 e ;

P[t_,e_,0]=L[t,e]/(1-e)^2/(t-N[F[e,1],16])/(t-N[F[e,2],16]) ;
P[t_,e_,1]=L[t,e]/(1-e)^2/(t-N[F[e,0],16])/(t-N[F[e,2],16]) ;
P[t_,e_,2]=L[t,e]/(1-e)^2/(t-N[F[e,0],16])/(t-N[F[e,1],16]) ;

a[t_,e_,x_,i_,n_]=t P[t,e,i]^n Exp[t(x-n)] ;

Dt[a[t,e,x,0,1],{t,0},Constants->{e,x}];

r[t_,e_,x_,0,1]=%;

Dt[a[t,e,x,1,1],{t,0},Constants->{e,x}];

r[t_,e_,x_,1,1]=%;

Dt[a[t,e,x,2,1],{t,0},Constants->{e,x}];

r[t_,e_,x_,2,1]=%;


Dt[a[t,e,x,0,2],{t,1},Constants->{e,x}];

r[t_,e_,x_,0,2]=%;

Dt[a[t,e,x,1,2],{t,1},Constants->{e,x}];

r[t_,e_,x_,1,2]=%;

Dt[a[t,e,x,2,2],{t,1},Constants->{e,x}];

r[t_,e_,x_,2,2]=%;


Dt[a[t,e,x,0,3],{t,2},Constants->{e,x}];

r[t_,e_,x_,0,3]=%;

Dt[a[t,e,x,1,3],{t,2},Constants->{e,x}];

r[t_,e_,x_,1,3]=%;
```

```
Dt[a[t,e,x,2,3],{t,2},Constants->{e,x}];

r[t_,e_,x_,2,3]=%;


Dt[a[t,e,x,0,4],{t,3},Constants->{e,x}];

r[t_,e_,x_,0,4]=%;

Dt[a[t,e,x,1,4],{t,3},Constants->{e,x}];

r[t_,e_,x_,1,4]=%;

Dt[a[t,e,x,2,4],{t,3},Constants->{e,x}];

r[t_,e_,x_,2,4]=%;


g[e_,x_,1]=(1/x) *
(r[N[F[e,0],16],e,x,0,1]+
 r[N[F[e,1],16],e,x,1,1]+
 r[N[F[e,2],16],e,x,2,1]);

g[e_,x_,2]=(1/x) (-12 e)*
(r[N[F[e,0],16],e,x,0,2]+
 r[N[F[e,1],16],e,x,1,2]+
 r[N[F[e,2],16],e,x,2,2]);

g[e_,x_,3]=(1/x) (-12 e)^2/Factorial[2]*
(r[N[F[e,0],16],e,x,0,3]+
 r[N[F[e,1],16],e,x,1,3]+
 r[N[F[e,2],16],e,x,2,3]);

g[e_,x_,4]=(1/x) (-12 e)^3/Factorial[3]*
(r[N[F[e,0],16],e,x,0,4]+
 r[N[F[e,1],16],e,x,1,4]+
 r[N[F[e,2],16],e,x,2,4]);

PY[e_,x_] = If [ x<1 ,0,Re[N[g[e,x,1],16]] ] +
            If [ x<=2,0,Re[N[g[e,x,2],16]] ] +
            If [ x<=3,0,Re[N[g[e,x,3],16]] ] +
            If [ x<=4,0,Re[N[g[e,x,4],16]] ] ;

a1[e_]=(1+2*e)^2/(1-e)^4 ;
c1[e_]=-2*e*(1+e/2)*(1+2*e)/(1-e)^4 ;
c2[e_]=e*a1[e]/2 ;

Integrate[t*(a1[e]*(t-x)+c1[e]+c2[e]),{t,0,x}] ;
```

```
int1[e_,x_]=% ;

Integrate[t*(x+1-t)^3*(c1[e]+c2[e]*(x+1-t)),
{t,x,1}] ;
int2[e_,x_]=% ;

HAB[e_,x_]:=If[x<0,0,N[(1+2*e)/(1-e)^2 -
(3/5)*e*(10-2*e+e^2)/(1+2*e)*a1[e]*x +
(e/5)*(3*c1[e]+2*c2[e]),16] + N[12*e,16] * N[
If[x<=1,-int1[e,x],
-NIntegrate[t*(a1[e]*(t-x)+c1[e]+c2[e]),
{t,0,1}] +
NIntegrate[t*(a1[e]*(t-x)+c1[e]+c2[e])*(PY[e,t]-1),
{t,1,x}] ] +
If[x<1,-int2[e,x] +
NIntegrate[t*(x+1-t)^3*(c1[e]+c2[e]*(x+1-t))*(PY[e,t]-1),
{t,1,x+1}] ,
NIntegrate[t*(x+1-t)^3*(c1[e]+c2[e]*(x+1-t))*(PY[e,t]-1),
{t,x,x+1}] ],16] ] ;

HAINT[e_, z_] := 6/*e*NIntegrate[HAB[e, x], {x, 0, z}] ;

NUVOL[e_, y_] := 6*
    e*NIntegrate[(0.25 - x^2)*HAB[e, y + x - 0.5], {x, -0.5, +0.5}, {
              PrecisionGoal -> 3}]

ITT (*approximation for HAB*)= Interpolation[{{0.01, 0, 1.04071}, {0.01, 0.1,
1.03443}, {
    0.01, 0.2, 1.02831}, {0.01, 0.3, 1.02245}, {0.01, 0.4, 1.017}, {
        0.01, 0.5, 1.01208}, {0.01, 0.6, 1.00781}, {0.01, 0.7, 1.0043}, {
      0.01, 0.8, 1.00169}, {0.01, 0.9, 1.0001}, {
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```

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        0.595674}, {0.6, 3.4, 0.526856}, {0.6, 3.5, 0.502061}, {0.6, 3.6,
      0.531445}, {0.6, 3.7, 0.59356}, {0.6, 3.8, 0.65316}, {0.6,
        3.9, 0.681664}, {0.6, 4., 0.668564}}]

HAINTAPPROX[e_, z_] := (* Approximation *)
6/*e*NIntegrate[If[x < 0, 0, ITT[e, x]], {x, 0, z}]

HABNUAPPROX[e_, al_, z_] := (* Approximation *) 6/*e*NIntegrate[If[x < 0,
  0, If[x > 4,
    1, ITT[e, x]]]*Exp[-3*(1 +
```

```
1/Cos[al])*NIntegrate[If[y > 4, e, NUIT[
  e, y]], {y, 0, x + (1 - Cos[al/2])/2}]], {x, 0, z}]
```

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