Spectral integrals for blackbody radiation

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1 Blackbody spectral distribution

Planck's (1899) spectral distribution of an equilibrium blackbody at temperature T versus wavelength λ is

$$I_{\lambda,b} = \frac{2hc^2}{\lambda^5\left[\exp\left(\frac{hc/\lambda}{kT}\right) - 1\right]}, \tag{1}$$

where $h \simeq 6.6\ 10^{-34}$ J.s is Planck's constant, $k \simeq 1.4\ 10^{-23}$ J/°K is Boltzmann's constant, and $c \simeq 3\ 10^8$ m/s is the speed of light *in vacuo*. Defining the dimensionless photon energy $E^\dagger \equiv (hc/\lambda)/(kT)$, the differentiation of Eq. (1) indicates that $I_{\lambda,b}$ peaks at values of $E^\dagger$ satisfying

$$E^\dagger \exp E^\dagger = 5[\exp E^\dagger - 1]. \tag{2}$$

Because $\exp E^\dagger \gg 1$, the solution to Eq. (2) is $E^\dagger \simeq 5$, which defines the locus of the peaks of Planck's distribution. This simple solution is called "Wien's displacement law" [1].

Because the blackbody has isotropic emission (i.e., independent of direction), its total radiative power $\dot{q}$ emitted per unit area A is

$$E = \frac{d\dot{q}}{dA} = \int_0^\infty \pi I_{\lambda,b} d\lambda = \frac{2\pi^5 k^4}{15c^2h^3}T^4, \tag{3}$$

where the group $2\pi^5k^4/(15c^2h^3) \equiv \sigma \simeq 5.7\ 10^{-8}$ W/m^2 °K^4 is called the Stefan-Boltzmann constant. Incropera, et al. [1] report this constant as $\sigma = (\pi^5/15)(c_1/c_2^4)$, where $c_1 \equiv 2hc^2$ and $c_2 \equiv hc/k$.

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2 Band emission

In some circumstances, it is useful to evaluate the blackbody radiative power emitted within a given band of wavelength $\lambda_1 \leqslant \lambda \leqslant \lambda_2$, see Incropera, et al., section 12.3.4 p. 739 [1]. To that end, it is convenient to define Planck's radiation function $P(\lambda^\dagger)$ such that

$$\frac{E_{\lambda,b}}{\sigma T^5} \equiv \Big(\frac{k}{hc}\Big) P(\lambda^\dagger) = \frac{\pi I_{\lambda,b}}{\sigma T^5}, \tag{4}$$

where the dimensionless wavelength is the inverse of $E^\dagger$,

$$\lambda^\dagger \equiv \frac{kT\lambda}{hc} = \frac{1}{E^\dagger}. \tag{5}$$

Planck's radiation function

$$P(\lambda^\dagger) \equiv \Big(\frac{15}{\pi^4}\Big) \frac{1}{\lambda^{\dagger 5}\Big(\exp(1/\lambda^\dagger) - 1\Big)} \tag{6}$$

is normalized i.e.,

$$\int_0^\infty P(\lambda^\dagger) d\lambda^\dagger = 1, \tag{7}$$

and its properties are summarized at `mathworld.wolfram.com/PlancksRadiationFunction.html`. This function is a "marginal" distribution in statistics' lingo. Its integral is the "cumulative" distribution for all dimensionless wavelengths below $\lambda^\dagger$

$$F(\lambda^\dagger) \equiv \int_0^{\lambda^\dagger} P(\iota) d\iota, \tag{8}$$

where ι is a dummy variable of integration. Because P is normalized according to Eq. (7), the function F asymptotes to unity as $\lambda^\dagger \to \infty$. Also, it obviously vanishes at $\lambda^\dagger = 0$. A convenient least-squares fit is

$$F(\lambda^\dagger) \simeq \begin{cases} 0 & \text{for } 0 \leqslant \lambda^\dagger < 0.092 \\ (\lambda^{\dagger 2} - 0.1655\lambda^\dagger + 0.00694)/(\lambda^{\dagger 2} - 0.181\lambda^\dagger + 0.0526) & \text{for } 0.092 \leqslant \lambda^\dagger < 2.05 \\ 1 & \text{for } \lambda^\dagger \geqslant 2.05. \end{cases} \tag{9}$$

Using the change of variable from λ to $\lambda^\dagger$ in the integral (8), it is relatively easy to show that the power per unit surface contained in the wavelength range $\lambda_1 \leqslant \lambda \leqslant \lambda_2$ is

$$\int_{\lambda_1}^{\lambda_2} E_{\lambda,b} d\lambda = \int_{\lambda_1}^{\lambda_2} \pi I_{\lambda,b} d\lambda = \sigma T^4 [F(\lambda_2^\dagger) - F(\lambda_1^\dagger)]. \tag{10}$$

Some may find this expression and the fit of Eq. (9) easier to use than Table 12.1 p. 720 of Incropera, et al. [1].

References

[1] F.P. Incropera, D.P. Dewitt, T.L. Bergman and A.S. Lavine. *Fundamentals of Heat and Mass Transfer, Sixth Edition*, Wiley, NY (2007).